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Spectrum structure and coherent state of the two-particle Calogero–Sutherland model: an application of the pseudo-angular-momentum operator method

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Abstract

The generality of the Calogero–Sutherland model (CSM) is studied with the aid of its several variations. The CSM is then solved by a new approach, the pseudo-angular-momentum operator method. The symmetry in spectrum space and valid region of the parameter are discussed. Analytic expressions of the eigenstate are obtained. The coherent states of the CSM are also discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In a centre-of-mass system (CMS), the stationary Schrödinger equation of the Calogero–Sutherland model (CSM) with two particles in one dimension is written as [1, 2]

$$H\psi(x) = E\psi(x), \qquad H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{g}{x^2}\right), \qquad E > 0, \quad g \geqslant -\frac{1}{4}. \tag{1}$$

This problem was raised many years ago [3–5] but has not been studied thoroughly. In fact, [1] studied the three-body problem of the one-dimensional identical particles. It discussed the different statistical types (Bose, Fermi or Boltzmann statistics) of the three-body problem in detail and further extended this model to the N identical particles case

$$H = \frac{1}{2}\left[-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} x_i^2 + g\sum_{j<i}^{N}\frac{1}{(x_i - x_j)^2}\right]. \tag{2}$$

Reference [6] classified equation (2) to the symmetry group $SU(N)$ and obtained the expressions of generators via the symmetric analysis of the group $SU(2)$. Paper [7] named equation (1) the isotonic oscillator, and obtained the general creation operator and general annihilation operator by factorization, and the shifter operator thus obtained the eigenfunctions of equation (1). References [8, 9] solved the eigenvalue problem of equation (1) as the special case of the group $SU(1,\ 1)$ and provided its coherent state. Other articles which studied the eigenvalue problem and coherent state of equation (1) include [10–12]. The two-particle CSM (1) is also referred to as B1-type Calogero model in Olshanetsky–Perelomov's classification [14].

In this paper, first we studied the generality of equation (1) with the aid of its several variations. Then, equation (1) is solved by the pseudo-angular-momentum operator method. The symmetry in spectrum space of CSM is discussed and the analytical expressions of the eigenstate are derived. Finally, the coherent state of the CSM is also discussed.

2. Several variations of the CSM

The aim of discussing several variations of the CSM is to illustrate its generality.

2.1. The radical equation of the three-dimensional isotropic oscillator

The radical equation of the three-dimensional isotropic oscillator in the CMS is

$$\begin{aligned} &\frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}+x^2+\frac{l(l+1)}{x^2}\right]R(x)=\lambda R(x),\\ &x=\alpha r,\quad \alpha=\sqrt{\frac{\mu\omega}{\hbar}},\quad \lambda=\frac{E}{\hbar\omega},\quad 0\leqslant x<\infty,\quad l=1,2,\ldots. \end{aligned} \tag{3}$$

Here E, μ and ω are the energy, reduced mass and angular frequency, respectively. Let $R(x)=x^{-1}\psi(x)$, then we have the equation

$$\frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x^2+\frac{l(l+1)}{x^2}\right]\psi(x)=\lambda\psi(x). \tag{4}$$

This is exactly equation (1) with $g=l\,(l-1)$.

2.2. The radical equation of hydrogen-like atom in spherical coordinates

The radical equation of a hydrogen-like atom in the CMS is

$$\begin{aligned} &\frac{1}{2}\left[-\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}+\frac{1}{4}\xi+\frac{l(l+1)}{\xi}\right]R(\xi)=\beta R(\xi);\\ &\xi=\alpha\gamma,\quad \alpha=\sqrt{-8\mu E/\hbar^2},\quad E<0,\quad \beta=\sqrt{-\mu z e_s^4/2\hbar^2 E}>0,\quad e_s^2=e^2/4\pi\varepsilon_0. \end{aligned} \tag{5}$$

Here μ, z, e and ε_0 are the reduced mass, proton number of the atomic nucleus, the absolute value of the electron charge and permittivity of the vacuum, respectively. Let $\xi=x^2$ in equation (5), then we have the equation

$$\frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{3}{x}\frac{\mathrm{d}}{\mathrm{d}x}+x^2+\frac{4l(l+1)}{x}\right]R(x)=2\beta R(x).$$

Let $R(x) = x^{-3/2}\psi(x)$, then we have

$$\frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2 + \frac{4l(l+1)+3/4}{x}\right]R(x) = 2\beta R(x). \tag{6}$$

This is the same as equation (1) with $g = 4l(l+1) + 3/4$, $E = 2\beta$.

2.3. The equation of hydrogen-like atom in parabolic coordinates

Select coordinates as (ξ, η, ϕ) in the radical equation of hydrogen-like atom in the CSM, where

$$\xi = \alpha\gamma(1 - \cos\theta), \qquad \eta = \alpha\gamma(1 + \cos\theta), \qquad \phi = \phi. \tag{7}$$

By separation of variables

$$\varphi(r,\theta,\phi) = f(\xi)f(\eta)\Phi(\phi), \qquad \Phi(\phi) = \frac{1}{\sqrt{2\pi}}\,e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots,$$

we obtain two equations of $f(\xi)$ and $g(\eta)$ of identical form

$$\begin{aligned} &\left(-\frac{d}{d\xi}\xi\frac{d}{d\xi} + \frac{1}{4}\xi + \frac{1}{4}\frac{m^2}{\xi}\right)f(\xi) = \gamma f(\xi), \\ &\left(-\frac{d}{d\eta}\eta\frac{d}{d\eta} + \frac{1}{4}\eta + \frac{1}{4}\frac{m^2}{\eta}\right)g(\eta) = \gamma_2 g(\eta); \quad \gamma_1 + \gamma_2 = \lambda. \end{aligned} \tag{8}$$

We only discuss one of them, the equation of $f(\xi)$

$$\left(-\frac{d}{d\xi}\xi\frac{d}{d\xi} + \frac{1}{4}\xi + \frac{1}{4}\frac{m^2}{\xi}\right)f(\xi) = \gamma f(\xi). \tag{9}$$

Let $\xi = x^2$, then we have

$$\frac{1}{2}\left(-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + x^2 + \frac{m^2}{x^2}\right)f(x) = 2\gamma f(x). \tag{10}$$

Let $f(\xi) = x^{-1/2}\psi(x)$, then we have

$$\frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{m^2 - 1/4}{x^2}\right)\psi(x) = 2\gamma\psi(x). \tag{11}$$

This is the same as equation (1) with $g = m^2 - 1/4$ and $E = 2\gamma$.

2.4. The CSM of the one-dimensional three-body problem

In the one-dimensional three-body problem proposed in [1], the quasi-radical equation in the CSM is

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{3}{8}\omega r^2 + \frac{b_l^2}{r^2}\right)R = ER, \qquad l = 0, 1, 2, \ldots. \tag{12}$$

Equation (12) is obtained via the coordinate transform and separation of variables. Actually, this equation is of the same form as equation (10). Let $x = \sqrt[4]{3\omega/8}\,r$ and $R = x^{-1/2}\psi(x)$, then we have the equation

$$\frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{b_l^2 - 1/4}{x^2}\right)\psi(x) = \lambda\psi(x), \qquad \lambda = \sqrt{\frac{2}{3\omega}}E. \tag{13}$$

This is the same as equation (1) with $g = b_l^2/2 - 1/4$.

2.5. The s state equation of Morse potential

The s state equation of Morse potential is

$$\frac{1}{2}\left(-\frac{d^2}{dr^2}+D\,e^{-2\tau r}-2D\,e^{-\tau r}\right)R(r)=ER(r), \qquad -D<E<0. \tag{14}$$

Here D and E are positive real constants. Let

$$x=\sqrt{\lambda_0}\,e^{-\tau r/2}, \qquad \lambda_0=\frac{2\sqrt{D}}{\tau} \qquad \text{or} \qquad D=\frac{\tau^2}{4}\lambda_0^2. \tag{15}$$

We have the equation

$$\left(-\frac{d^2}{dx^2}-\frac{1}{x}\frac{d}{dx}+x^2-\frac{8E}{\tau^2}\frac{1}{x^2}\right)R=2\lambda_0 R. \tag{16}$$

Let $R=x^{-1/2}\psi(x)$, then we have

$$\frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2+\frac{-8E/\tau^2-3/4}{x^2}\right)\psi(x)=\lambda_0\psi(x). \tag{17}$$

Let

$$g=-\frac{8E}{\tau^2}-\frac{3}{4}, \tag{18}$$

then equation (17) is converted to equation (1). But here $g=g(E)$, which means the energy is included in g.

The several variations of the CSM provided in this section show that the CSM is quite a general model. Each variation corresponds to a definite value of the parameter g. Thus, we discuss equation (1) with the general case of g.

We write all these problems in the CSM to emphasize those problems which essentially are two-particle problems.

3. Solving the CSM equation by the pseudo-angular-momentum method

We choose a new method to solve equation (1). First, we construct the operators

$$\begin{aligned} A_1&=\frac{i}{4}\left(-\frac{d^2}{dx^2}-x^2+\frac{g}{x^2}\right), \qquad & A_2&=\frac{1}{4}\left(\frac{d}{dx}x+x\frac{d}{dx}\right)=\frac{1}{2}\left(\frac{d}{dx}x-\frac{1}{2}\right),\\ A_3&=\frac{1}{4}\left(-\frac{d^2}{dx^2}+x^2+\frac{g}{x^2}\right); \qquad & A^2&\equiv A_1^2+A_2^2+A_3^2. \end{aligned} \tag{19}$$

It is obvious that

$$A_3=\tfrac{1}{2}H, \qquad A_1=i\left(A_3-\tfrac{1}{2}x^2\right). \tag{20}$$

And we can prove

$$\begin{aligned} &[A_1,A_2]=iA_3, \qquad [A_2,A_3]=iA_1, \qquad [A_3,A_1]=iA_2,\\ &[A^2,A_k]=0, \qquad k=1,2,3. \end{aligned} \tag{21}$$

Thus, we have

$$A_3\psi(x)=\lambda\psi(x), \qquad E=2\lambda, \quad \lambda>0. \tag{22}$$

Due to equations (21) and (22), we called our solving method the pseudo-angular-momentum operator method. We construct the general creation operator and general annihilation operator

$$A^+\equiv -iA_1+A_2, \qquad A\equiv -iA_1-A_2. \tag{23}$$

Then, we can rewrite equation (22) as

$$A^{+}A\psi_{\lambda} = [\lambda(\lambda-1) - A^2]\psi_{\lambda}, \qquad AA^{+}\psi_{\lambda} = [\lambda(\lambda+1) - A^2]\psi_{\lambda}. \tag{24}$$

Solving equations (19) and (20), we have

$$A^2 = \tfrac{1}{4}\left(g - \tfrac{3}{4}\right). \tag{25}$$

It is obvious that the recurrence relation of the eigenfunction is

$$A\psi_{\lambda} = C_1\psi_{\lambda-1}, \qquad A^{+}\psi_{\lambda} = C_2\psi_{\lambda+1}. \tag{26}$$

To make sure ψ_{λ} and $\psi_{\lambda\pm1}$ are normalized, we use the awaiting defined constants C_1 and C_2 in equation (26). From equation (26), we have the eigenvalue

$$\lambda = \lambda_0, \lambda_0+1, \lambda_0+2, \ldots, \qquad \lambda_0 > 0. \tag{27}$$

For $\lambda > 0$, and the lower limit of λ is λ_0, so we have $\lambda_0 > 0$. The corresponding state function is $\psi_{\lambda_0 l} = \psi_{0l}$, so we have

$$A\psi_{0l} = 0. \tag{28}$$

4. Symmetry in the spectrum space

To the left multiple A^{+} in equation (28), then we have

$$A^{+}A\psi_{0l} = 0. \tag{29}$$

To consider the first part of (24) includes $A^{+}A\psi_{0l} = [\lambda_0(\lambda_0-1) - A^2]\psi_{0l}$, so

$$[\lambda_0(\lambda_0-1) - A^2]\psi_{0l} = 0. \tag{30}$$

For ψ_{0l} to be not always equal to zero, the necessary condition for equation (30) to be valid is $\lambda_0\,(\lambda_0-1) - A^2 = 0$, so we obtain

$$\lambda_0 = \frac{1}{2}\left(1 \pm \sqrt{g + \frac{1}{4}}\right). \tag{31}$$

Note that this expression implies the origin of the $g \geqslant -1/4$ in equation (1). The explanation of this point is not very clear in some references; however, it is clear in others [1, 5]. And the explanation is not correct in some references, for example in [7], $g \geqslant -1/2$.

So we obtain the eigenvalue of A_3

$$\lambda = n + \frac{1}{2}\left(1 \pm \sqrt{g + \frac{1}{4}}\right), \qquad n = 0, 1, 2, \ldots. \tag{32}$$

The energy spectrum of equation (1) is

$$E = 2\lambda = 2n + 1 \pm \sqrt{g + \frac{1}{4}}. \tag{33}$$

Figure 1 gives the energy spectrum, i.e. the relation between E and g. Most references only obtained the dashed line part of figure 1; some obtained the solid line part. All the cases of figure 1 have not been discussed in detail until now.

When $g = 0$, equation (1) converted to the standard linear oscillator. The energy level of the standard linear oscillator is

$$E = \begin{cases} 1/2, 5/2, 9/2, \ldots \\ 3/2, 7/2, 11/2, \ldots. \end{cases} \tag{34}$$

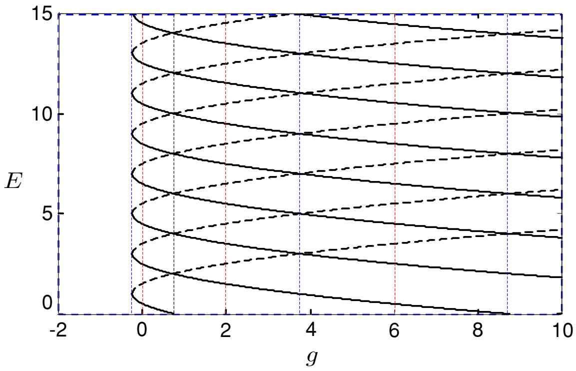

Figure 1. The relation between g and E.

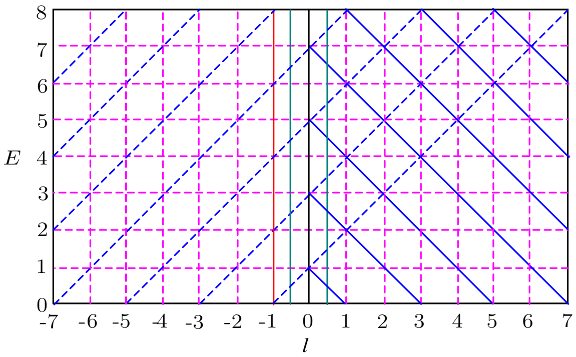

Figure 2. To fold the right part of the solid line to the left.

First, figure 1 shows that the base state varies with g. Also, there might be some accidental degeneracy as indicated by those points of intersection of the curves. Let us make a thorough analysis of this energy spectrum.

For the convenience of the analysis, we rewrite g as

$$g = l^2 - \tfrac{1}{4}. \tag{35}$$

Equation (1) can be rewritten as

$$\frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{l^2 - 1/4}{x^2}\right)\psi(x) = E\psi(x), \qquad l > -1, \tag{36}$$

where l is a continuous real parameter. It seems that we can choose $-\infty \leqslant l \leqslant +\infty$ or we can choose $l \geqslant 0$. But we choose $l > -1$ here; the reason is given below. The right part of figure 2 ($l \geqslant 0$) corresponds to figure 1. By folding the right part of the solid line to the left, taken $l = 0$ as the axis, we can obtain figure 3 ($-\infty \leqslant l \leqslant +\infty$).

We can see from figure 3 that the zones such as

$$(-1,\ 1], \qquad (-3,\ -1], \qquad (-5,\ -3], \ldots \tag{37}$$

are of the shifter invariance and the zones such as $(-3, -1]$, $(-5, -3]$, $(-7, -5]$, ... are redundant. The useful part of the energy spectrum is shown in figure 4. The reason why we choose $l > -1$ instead of $l \geqslant -1$ will be given later.

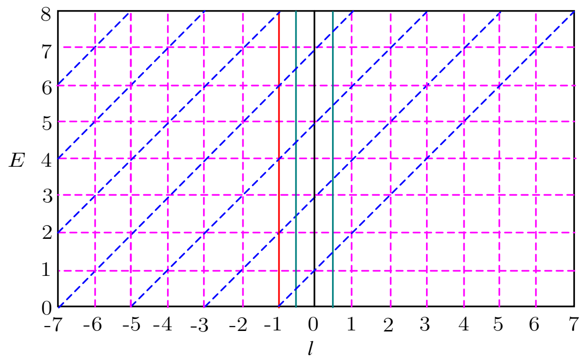

Figure 3. Relation between E and l.

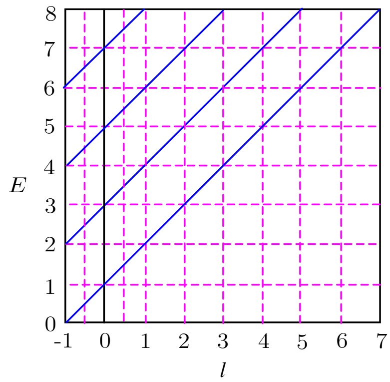

Figure 4. The useful part of the relation between E and l.

For convenience, we rewrite equations (33), (22), (24), (26), (28) and (23) as

$$E_{nl} = 2n + 1 + l, \qquad l > -1, \tag{38}$$

$$A_3\psi_{nl}(x) = \left(n + \frac{l}{2} + \frac{1}{2}\right)\psi_{nl}(x), \tag{39}$$

$$A^+A\psi_{nl}(x) = n(n+l)\psi_{nl}(x), \qquad AA^+\psi_{nl}(x) = (n+1)(n+l+1)\psi_{nl}(x), \tag{40}$$

$$A\psi_{nl}(x) = C_1\psi_{n-1,l}(x), \qquad A^+\psi_{nl}(x) = C_2\psi_{n+1,l}(x), \tag{41}$$

$$A\psi_{0l}(x) = 0, \tag{42}$$

$$A = A_3 + \frac{1}{4} - \frac{1}{2}x^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}x, \qquad A^+ = A_3 + \frac{1}{4} - \frac{1}{2}x^2 + \frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{43}$$

Operators A and A^+ can act on any state functions and also on the eigenstate. If acting on the eigenstate $\psi_{nl}(x)$, they can be written as

$$A_{nl} = \frac{1}{2}\left(2n + l + \frac{3}{2} - x^2 - \frac{\mathrm{d}}{\mathrm{d}x}x\right), \qquad A^+_{nl} = \frac{1}{2}\left(2n + l + \frac{3}{2} - x^2 + \frac{\mathrm{d}}{\mathrm{d}x}x\right) \tag{44}$$

directly.

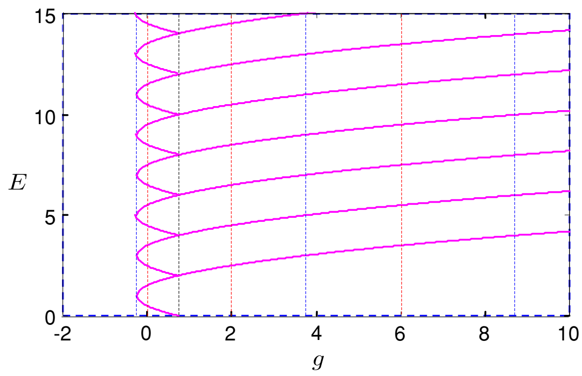

Figure 5. The useful part of the relation between E and g.

5. The set of normalized eigenfunctions

The base state can be derived from equation (28) or (42) directly. To substitute the first expression of (43) into (42), we have

$$\frac{1}{2}\left(l+\frac{3}{2}-x^2-\frac{\mathrm{d}}{\mathrm{d}x}x\right)\psi_{0l}(x)=0,$$

the solution is

$$\psi_{0l}(x)=N_{0l}x^{l+1/2}e^{-x^2/2}. \tag{45}$$

Note the integration limits and integration weight when computing the normalized constant N_{0l}. For the various problems in section 2, they are different. For example, the integration limit is $0<x<\infty$ for the radial equation of the three-dimensional isotropic oscillator and the weight is x^2. Here the integration limit is $0<x<\infty$ and the weight is 1. The solution of expression (45) is

$$N_{0l}=\sqrt{\frac{2}{\Gamma(l+1)}},\qquad l>-1 \tag{46}$$

For Γ function, we have $\Gamma(s+1)=s\Gamma(s)$. To make $s>0$, we must guarantee $l>-1$, $E_{0l}=l+1>0$, when $s\to 0^+$, $\Gamma(s)\to\infty$. This is the origin of the $l>-1$ in equation (36). And it is also the reason for $E>0$ but does not include $E=0$.

Thus, every curve does not include the lowest point, the point $l=-1$. We draw the corresponding $E(g)$ relationship of figure 4 in figure 5. Also, every curve in figure 5 does not include the lowest point. That means those curves do not intersect, thus there is no accidental degeneracy.

The recurrence relations of the normalized eigenfunction $\psi_{nl}(x)$ can be easily derived from equations (40) and (41)

$$A\psi_{nl}(x)=\sqrt{n(n+l)}\psi_{n-1,l}(x),\qquad A^+\psi_{nl}(x)=\sqrt{(n+1)(n+l+1)}\psi_{n+1,l}(x). \tag{47}$$

So we have

$$\psi_{nl}(x)=\frac{A^+}{\sqrt{n(n+l)}}\psi_{n-1,l}(x), \tag{48}$$

and then

$$\psi_{nl}(x)=\sqrt{\frac{2}{n!\Gamma(n+l+1)}}(A^+)^n x^{(l+1)/2}\,\mathrm{e}^{-x^2/2},\qquad l>-1. \tag{49}$$

Note that from the Γ-function in the denominator of expression (49)

$$\Gamma(n+l+1) = (n+l)(n+l-1)\cdots l\,\Gamma(l+1),$$

we can see that $l > -1$ is the necessary condition. Again, we confirm that figure 4 is the valid part of the energy spectrum. In fact, if we extend the first excited state $\psi_{1l}(x)$ of figure 3 to the zone $(-3, -1]$, it should be the base state in this zone. But it does not fulfil equation (42) $A\psi_{1l}(x) \neq 0\,(-3 < l \leqslant -1)$.

It can be proved that the operator (44) can be rewritten as

$$A_{nl} = -\frac{1}{2}x^{2n+l+3/2}\mathrm{e}^{-x^2/2}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{x^2/2}x^{-2n-l-1/2}, \tag{50}$$

$$A^+_{nl} = \frac{1}{2}x^{-2n-l-1/2}\,\mathrm{e}^{x^2/2}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{-x^2/2}x^{2n+l+3/2}. \tag{51}$$

By substituting (51) into (49), we obtain the analytical expression of the normalized engienfunciton

$$\psi_{nl}(x) = \frac{1}{2^n}\sqrt{\frac{2}{n!\Gamma(n+l+1)}}x^{\frac{-2n+l+1}{2}}\,\mathrm{e}^{-\frac{x^2}{2}}\left[x^{-(2l-1)}\,\mathrm{e}^{x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}x^3\right)^n\mathrm{e}^{-x^2}x^{2l-1}\right], \tag{52}$$

This is the analytical expression, while the eigenfuncitons given in other references are in series form [4–13]. The expression inside the square brackets in (52) is the polynomial part of the eigenfunction, which is the Rodrigues formula for the Laguerre polynomials. Reference [15] obtained the same Laguerre polynomials. The calculation of eigenfunction is done through another arising operator method in [18].

The base states for $l = -1/2$, $l = 1/2$ and $l = 0$ are

$$\psi_{0,-1/2}(x) = \sqrt{\frac{2}{\sqrt{\pi}}}\,\mathrm{e}^{-x^2/2}, \qquad \psi_{0,1/2}(x) = \sqrt{\frac{4}{\sqrt{\pi}}}x\,\mathrm{e}^{-x^2/2}, \qquad \psi_{0,0}(x) = \sqrt{2}x^{1/2}\,\mathrm{e}^{-x^2/2} \tag{53}$$

Here, $\psi_{0,-1/2}(x)$ and $\psi_{0,1/2}(x)$ are the base state and first excited state of the classical one-dimensional oscillator, respectively. Note that we choose $0 < x < \infty$ instead of $-\infty < x < \infty$; the reason is the factor $\sqrt{2}$ in the normalized constant. The functions $\psi_{0,-1/2}(x)$ and $\psi_{0,1/2}(x)$ both are the base states with the different value of the parameter l in this paper. If we choose $l = 0$, it corresponds to the last expression of equation (53).

We must emphasize that the CSM problem is the problem of the two particles in the CSM; it is not the one-photon problem. In fact, even the one-dimensional linear oscillator is not a one-photon problem. Thus, to take $\psi_{0,1/2}(x)$ as the first excited state is questionable; it should be taken as the base state with the parameter $l = 1/2$. We focus on discussing the characteristics of the CSM in the following chapters.

6. Parametrized two-particle system

Equation (43) can be factorized to

$$\begin{aligned} A &= -\frac{1}{4}\left(x + \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l-1/2}{x}\right)\left(x + \frac{\mathrm{d}}{\mathrm{d}x} + \frac{l-1/2}{x}\right) \\ &= -\frac{1}{4}\left(x + \frac{\mathrm{d}}{\mathrm{d}x} + \frac{l+1/2}{x}\right)\left(x + \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l+1/2}{x}\right), \end{aligned} \tag{54}$$

$$A^+ = -\frac{1}{4}\left(x - \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l+1/2}{x}\right)\left(x - \frac{\mathrm{d}}{\mathrm{d}x} + \frac{l+1/2}{x}\right)$$
$$= -\frac{1}{4}\left(x - \frac{\mathrm{d}}{\mathrm{d}x} + \frac{l-1/2}{x}\right)\left(x - \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l-1/2}{x}\right), \tag{55}$$

when $l = \pm 1/2$, we have

$$A = -\frac{1}{2}a^2, \qquad A^+ = -\frac{1}{2}(a^+)^2, \tag{56}$$

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right), \qquad a^+ = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{57}$$

where a and a^+ are the annihilation operator and creation operator of the normal linear oscillator.

Taking $l = -1/2$ in expression (49) and substituting it into the second part of expressions (56), we have

$$\psi_n^{\mathrm{e}}(x) = \psi_{n,-1/2}(x) = \frac{(a^+)^{2n}}{\sqrt{2^{2n}n!\Gamma(n+1/2)}}\sqrt{2}\,\mathrm{e}^{-x^2/2}, \tag{58}$$

where the $(-1)^n$ is included in the normalized factor. Using the Legendre formula of the Γ-function,

$$\sqrt{\pi}\Gamma(2n) = 2^{2n+1}\Gamma(n)\Gamma(n+1/2). \tag{59}$$

Expression (58) can be written as

$$\psi_n^{\mathrm{e}}(x) = \psi_{n,-1/2}(x) = \frac{a^+}{\sqrt{(2n)!}}\left(\sqrt{\frac{2}{\sqrt{\pi}}}\,\mathrm{e}^{-x^2/2}\right). \tag{60}$$

The factor in the last parentheses is the base state eigenvalue of the linear oscillator. Expression (60) is the even Fock state of the two particles. Taking $l = 1/2$ in expression (49) and substituting it into the second part of expressions (56), we have

$$\psi_n^{\mathrm{o}}(x) = \psi_{n,1/2}(x) = \frac{(a^+)^{2n}}{\sqrt{2^{2n}n!\Gamma(n+3/2)}}\sqrt{2}x\,\mathrm{e}^{-x^2/2}. \tag{61}$$

Note that $\sqrt{2}x\,\mathrm{e}^{-x^2/2} = a^+\mathrm{e}^{-x^2/2}$, using the expression (59) again, we have

$$\psi_n^{\mathrm{o}}(x) = \psi_{n,1/2}(x) = \frac{(a^+)^{2n+1}}{\sqrt{(2n+1)!}}\left(\sqrt{\frac{2}{\sqrt{\pi}}}\,\mathrm{e}^{-x^2/2}\right). \tag{62}$$

This is the odd Fock state of the two particles.

Expressions (60) and (62) inform us that the state given by the expression (49) is the two-particle Fock state controlled by the continuous variable parameter l. Expressions (60) and (62) are only the special case of it. Figure 4 provides the spectrum of the state. CSM describes a more general two-particle system. If A^+ and A are taken as the creation operator and annihilation operator, respectively, they operate on a pair of the particle instead of the single particle. The several systems given in section 2 are all the two-particle systems in the CSM. Incidentally, those particles can be identical or different. This is very important. When discussing the general case of the coherent state of the two-particle Fock state controlled by the continuous variable parameter l, it includes the even and odd coherent states of the two particles automatically.

7. Coherent state of two particles with parameter

The coherent state is the eigenstate of the annihilation operator A. By the definition of

$$A\psi_{zl}(x) = z\psi_{zl}(x), \tag{63}$$

where z is the complex parameter. Let

$$\psi_{zl}(x) = \sum_{n=0}^{\infty} C_n \psi_{nl}(x), \tag{64}$$

where C_n is the undetermined constant. Substituting expression (64) into the left part of (63) and considering the first part of equation (47), we have

$$\begin{aligned} A\psi_{zl}(x) &= \sum_{n=0}^{\infty} C_n A\psi_{nl}(x) = \sum_{n=0}^{\infty} C_n \sqrt{n(n+l)}\psi_{n-1,l}(x) \\ &= \sum_{n=0}^{\infty} C_{n+1}\sqrt{(n+1)(n+l+1)}\psi_{nl}(x), \qquad l > -1. \end{aligned} \tag{65}$$

Substituting expression (64) into the right part of (63), we have

$$z\psi_{zl}(x) = \sum_{n=0}^{\infty} C_n z\psi_{nl}(x). \tag{66}$$

By comparing equation (65) and (66), we have

$$C_n = \frac{z}{\sqrt{n(n+l)}} C_{n-1} = \frac{z^n\sqrt{\Gamma(l+1)}}{\sqrt{n!\Gamma(n+l+1)}} C_0. \tag{67}$$

Substituting it into expression (64), we have

$$\psi_{zl}(x) = C_0 \sum_{n=0}^{\infty} \frac{z^n\sqrt{\Gamma(l+1)}}{\sqrt{n!\Gamma(n+l+1)}} \psi_{nl}(x). \tag{68}$$

In expression (68), using the normalized condition, we obtain C_0

$$C_0 = \left[\sum_{n=0}^{\infty} \frac{|z|^{2n}\sqrt{\Gamma(l+1)}}{\sqrt{n!\Gamma(n+l+1)}}\right]^{-\frac{1}{2}}. \tag{69}$$

And the orthonormal of the eigenfunction,

$$\int \psi_{n'l}(x)\psi_{nl}(x)\,\mathrm{d}x = \delta_{n'n}, \tag{70}$$

is used in the derivation of equation (69). Note that $\psi_{nl}(x)$ and $\psi_{n'l}(x)$ correspond to the same l value. The coherent state (68) can be written as

$$\psi_{zl}(x) = C_0 \sum_{n=0}^{\infty} \frac{(zA^+)^n}{n!\Gamma(n+l+1)} x^{l+1/2}\,\mathrm{e}^{-x^2/2}. \tag{71}$$

We will prove that this coherent state will convert into the even and odd coherent states when taking parameter $l = \pm 1/2$. Substituting $l = -1/2$ into expressions (68) and (69), we have

$$\begin{aligned} \psi_z^{\mathrm{e}}(x) &= C_0^{\mathrm{e}} \sum_{n=0}^{\infty} \frac{z^n\Gamma(1/2)}{2^n n!\Gamma(n+3/2)} \psi_n^{\mathrm{e}}(x) = C_0^{\mathrm{e}} \sum_{n=0}^{\infty} \frac{(2z)^n}{\sqrt{(2n)!}} \psi_n^{\mathrm{e}}(x), \\ C_0^{\mathrm{e}} &= \left[\sum_{n=0}^{\infty} \frac{|2z|^{2n}}{(2n)!}\right]^{-1/2} = (\cosh|2z|)^{-1/2}, \qquad \psi_n^{\mathrm{e}}(x) = \psi_{n,-1/2}(x), \end{aligned} \tag{72}$$

$$\psi_z^{\mathrm{e}}(x) = (\cosh|2z|)^{-1/2} \sum_{n=0}^{\infty} \frac{(2z)^n}{\sqrt{(2n)!}} \psi_n^{\mathrm{e}}(x). \tag{73}$$

The Legendre formula of the Γ function (59) is used in the derivation. This is the so-called even coherent state of the Glauber coherent state.

For parameter $l = 1/2$, using the same method we have the even coherent states

$$\psi_z^{\mathrm{o}}(x) = (\sinh|2z|)^{-1/2} \sum_{n=0}^{\infty} \frac{(2z)^{n+1/2}}{\sqrt{(2n+1)!}} \psi_n^{\mathrm{o}}(x), \tag{74}$$

$$\psi_n^{\mathrm{o}}(x) = \psi_{n,1/2}(x). \tag{75}$$

Expression (74) is the even coherent state of the Glauber coherent state.

Since the coherent state (71) includes the even and odd coherent states of the Glauber coherent states (73) and (74), the expression (71) is a more general coherent state of two particles. Thus, the CSM is quite a general parametrized two-particle system.

8. Conclusions

In this paper, several variations of the CSM equation are given to show that the CSM equation is very general. Then, the eigenvalue set of the CSM equation is obtained using a new method called the pseudo-angular-momentum operator method. The symmetry in spectrum space is analysed and the valid region of parameter g or l is determined. The previous confusion is resolved. After that, the analytical expression of the eigenfunction set is derived. It has been pointed out that this set of the eigenfunction includes the even and odd Fock states of two particles by analysing the set of eigenfunction. Finally, the coherent state of the CSM system is provided and it has been proven that it includes the even and odd coherent states of the Glauber coherent state automatically. Thus, it shows that the CSM and its parameter are a more general two-particle system. The pair of particles can be identical or different. The several systems given in section 2 are all two-particle systems. Incidentally, this paper clarifies that the one-dimensional linear oscillator is not a one-particle problem; it is the special case of the CSM, and naturally a two-particle system.

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